OPTIMAL CONTROL OF A NONLINEAR MODEL OF ECONOMIC GROWTH

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Abstract. A nonlinear control model of a firm describing the change of production and accumulated R&D investment is investigated. An optimal control problem with R&D investment rate as a control parameter is solved. Optimal dynamics of economic growth of a firm versus the current cost of innovation is studied. It is analytically determined that dependent on the model parameters, the optimal control must be of one of the following types: a) piecewise constant with at most two switchings, b) piecewise constant with two switching and containing a singular arc. The intervals on which switching from regular to singular arcs occur are found numerically. Finally, optimal investment strategies and production activities are compared with econometric data of an actual firm.

1. Introduction. While models of economic growth have a long history in mathematical economics ([1]-[3]), they have recently received renewed interest ([4],[5]) due to improvements in existing models and refinements in the associated analytical techniques ([6]-[8]). This general intellectual interest finds expression in concurrent governmental sponsorship of research among the developed countries (e.g. Japan) with the expectation that accurate economic growth models will be of direct benefit to the management of their economic development ([8]-[10]).

There are many non-equivalent mathematical methods of measuring the effectiveness of economic growth. In this paper our objective is to maximize a functional of the total production assuming that the market is stable and that demand for the product always exists. The proposed model has never been analyzed before with this objective. An advantage of using such an objective is the ability to obtain the type of optimal solution analytically for any given model parameters.

The optimal solution of the problem described in Sections 2-3 of this paper is the unique solution of the boundary value problem of the Maximum Principle ([11]). The specifics of the model (Section 2) and properties of the boundary value problem of the Maximum Principle (Sections 3-4) allow us to reduce a very complex boundary value problem of four differential equations to a more manageable equivalent boundary value problem of two differential equations, boundary conditions

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and condition of the maximum. Original phase variables are expressed in terms of the variables of the new boundary value problem. (Section 5). The benefits of our analytical investigation conducted in Sections 2-5 in relation to a pure numerical calculation are: a warranty is provided for the obtained results, the type of optimal solution can be found for any given set of the model parameters, a complex boundary value problem is simplified to one of finite dimensional optimization, concrete guidance for the management of a firm is more accessible. Simulations with an application to the Sony Corporation are given in Section 6.

2. The model and its properties. Consider a nonlinear control system describing interactions between the production of the company, its technology stock, and R&D investments:

\[
\begin{align*}
\dot{x}_1(t) &= \phi_1 x_1(t) + \phi_2 \left( \frac{x_2(t)}{x_1(t)} \right)^\gamma x_1(t) - g_1 u(t) x_1(t), \\
\dot{x}_2(t) &= g_2 u(t) x_1(t), \quad t \in [0, T], \\
x_1(0) &= x_1^0, \quad x_2(0) = x_2^0, \quad x_1^0, x_2^0 > 0.
\end{align*}
\]

Here, \(x_1(t)\) is production, \(x_2(t)\) is the total technology stock, \(u(t)\) is R&D intensity, \(g_1\) is the discounted marginal productivity of technology (negative productivity is interpreted as a risk factor of technological investments), \(g_2\) is a coefficient of expenses for technology development, \(\gamma \in (0, 1)\) is an elasticity parameter of technology to production (the case \(\gamma = 1\) was investigated in [12]), \(\phi_1\) is the R&D contribution to increases in production, \(\phi_2\) is the non-R&D contribution to increases in production, and \(T\) is the end of the planning period.

For the model (1) \(x_1, x_2\) are variables and \(\phi_1, \phi_2, g_1, g_2, x_1^0, x_2^0, T\) are parameters. Function \(u(t)\) is the control function. Thus, the system (1) is a nonlinear control system. We call \(D(T)\) the control set, i.e. the set of all Lebesgue measurable functions, \(u(t)\), satisfying the inequality, \(0 < u_1(t) \leq u(t) \leq u_2\), for almost all \(t \in [0, T]\).

The following statement describes the property of variables \(x_1(t)\) and \(x_2(t)\) for the system (1).

**Theorem 1.** Let \(u(\cdot) \in D(T)\) be some control function. The components of the solution of the system (1) corresponding to \(u(t)\), \(x(t) = (x_1(t), x_2(t))^T\), satisfy the inequalities:

\[
0 < x_1(t) < M_1, \quad 0 < x_2(t) < M_2, \quad t \in [0, T].
\]

The symbol \(\top\) indicates the transpose. Values \(M_1\) and \(M_2\) are positive constants dependent on parameters of the system (1).

3. Optimal control problem. Since the market is stable and there is always a demand for the product, we assume that the firm is interested in maximizing its final production funds \(x_1(T)\) thereby yielding a maximal cumulative flow of profit until time \(t = T\). Therefore, an optimal control problem maximizing production for the system (1) can be written as

\[
J(u) = x_1(T) \to \max_{u(\cdot) \in D(T)};
\]

that is, find such an optimal R&D investment policy for which the total production until time \(T\) is maximized.

The existence of the optimal control \(u_*(t)\) and its corresponding optimal trajectory \(x_*(t) = (x_1^*(t), x_2^*(t))^T\), \(t \in [0, T]\) for the problem (1),(2) follows from [13].
In order to solve the optimal control problem (1), (2) we will apply the Pontryagin Maximum Principle ([11]). For the optimal control, \( u_*(t) \), and its corresponding optimal trajectory, \( x_*(t) \), there exists a non-trivial solution, \( \psi_* (t) = (\psi_1^* (t), \psi_2^* (t))^\top \), of the adjoint system,

\[
\begin{align*}
\dot{\psi}_1^*(t) &= -\left( \phi_1 + (1 - \gamma)\phi_2 \left( \frac{x_1^*(t)}{x_1^1(t)} \right)^\gamma - g_1 u_*(t) \right) \psi_1^*(t) - g_2 u_*(t) \psi_2^*(t), \\
\dot{\psi}_2^*(t) &= -\gamma \phi_2 \left( \frac{x_2^*(t)}{x_1^1(t)} \right)^{\gamma-1} \psi_1^*(t), \quad t \in [0, T], \\
\psi_1^*(T) &= 1, \quad \psi_2^*(T) = 0.
\end{align*}
\]

(3)

Here \( \psi_1^*(t) \) and \( \psi_2^*(t) \) are the so-called "shadow prices" of production and technology, respectively. The optimal control is given by

\[
u_* (t) = \begin{cases} u_1 & \text{if } L(t) < 0, \\ \forall u \in [u_1, u_2] & \text{if } L(t) = 0, \\ u_2 & \text{if } L(t) > 0, \end{cases}
\]

(4)

where \( L(t) = g_2 \psi_2^*(t) - g_1 \psi_1^*(t) \) is the switching function which determines the type of optimal control, \( u_*(t) \). Systems (1), (3) and relationship (4) form a two point boundary value problem of the Maximum Principle.

We will study this problem in depth. If the control \( \bar{u}(t) \) and trajectory \( \bar{x}(t) \) with adjoint function, \( \bar{\psi}(t) \) satisfy (1), (3) and (4), then \( \bar{u}(t) \) is called the extremal control, \( \bar{x}(t) \) is the extremal trajectory, and \( \bar{\psi}(t) \) is the corresponding solution of the adjoint system. Since the Pontryagin Maximum Principle is only the necessary condition of optimality, then the optimal control \( u_*(t) \) and optimal trajectory \( x_*(t) \) satisfy the boundary value problem of the Maximum Principle ([11]). Therefore it is incumbent on us to study the properties of the extremal control \( \bar{u}(t) \), extremal trajectory \( \bar{x}(t) \), and the corresponding solution \( \bar{\psi}(t) \), of the adjoint system (3).

**Theorem 2.** The components \( \tilde{\psi}_1(t) \) and \( \tilde{\psi}_2(t) \) of the adjoint function \( \tilde{\psi}(t) \) are positive i.e.

\[
\tilde{\psi}_1(t) > 0, \quad \tilde{\psi}_2(t) > 0, \quad t \in [0, T).
\]

The results of Theorem 1 and Theorem 2 allow us to establish the validity of the following statements.

**Theorem 3.** The extremal control \( \bar{u}(t) \) and extremal trajectory \( \bar{x}(t) \) provide the maximum of the functional \( J(u) \) given by (2).

**Theorem 4.** The boundary value problem of the Maximum Principle (1), (3), (4) has a unique solution.

Based on the results of Theorem 3 and Theorem 4, we formulate the important statement.

**Corollary 1.** For the optimal control problem (1), (2) there exists a unique optimal control \( u_*(t) \) and optimal trajectory \( x_*(t) \) that together with the solution \( \psi_*(t) \) of the adjoint system form a unique solution to the boundary value problem of the Maximum Principle (1), (3), (4).

Analyzing the switching function \( L(t) \) we obtain the following property of the optimal control \( u_*(t) \).

**Theorem 5.** There exists such \( \tau \in [0, T) \), that on the segment, \( (\tau, T] \), the optimal control \( u_*(t) \), takes value \( u_1 \).
4. Singular control. The results of Theorem 1 and Theorem 2 allow us to introduce the variables:
\[ z(t) = \frac{x_2^*(t)}{x_1^*(t)}, \eta(t) = \frac{\psi_2^*(t)}{\psi_1^*(t)}. \] (5)

Then the switching function \( L(t) \) from (4) becomes the expression \( (\eta(t) - \frac{g_1}{g_2}) \).

Now we consider an opportunity for the optimal control \( u_*(t) \) to contain singular arcs. Let us assume that the switching function \( L(t) \) is zero on the interval \( \Delta \in (0, T) \):
\[ \eta(t) - \frac{g_1}{g_2} = 0, \quad t \in \Delta, \] (6)

then the derivative of the switching function \( L(t) \) is also zero on the interval \( \Delta : \dot{\eta}(t) = 0, \quad t \in \Delta. \) To calculate the function \( \dot{\eta}(t) \) we use its representation from (5) and formulas from (4). Applying the equality (6), we obtain the relationship \( h_1(z(t)) = h_2(z(t)) \), where \( h_1(z) = g_1\phi_1 z^{1-\gamma} \) and \( h_2(z) = \gamma g_2\phi_2 - (1-\gamma)g_1\phi_2 z \). For \( z \geq 0 \) function \( h_2(z) \) is linear in \( z \) with a negative slope and \( h_1(z) \) is monotonically increasing. Then equation \( h_1(z) = h_2(z) \) has a unique solution \( \hat{z} \), such that \( \hat{z} \in (0, \frac{\gamma g_2}{(1-\gamma)g_1}) \). Therefore, we have \( z(t) = \hat{z} \) for all \( t \in \Delta \). It means that \( \hat{z} \) is the value of the function \( z(t) \) on the singular regime.

In order to find the value of singular control, we set the second derivative of the switching function \( L(t) \) to zero on the interval \( \Delta \). Hence, we find the value of singular control as
\[ \dot{u} = \frac{(\phi_1 + \phi_2\hat{z}^\gamma)\hat{z}}{g_1\hat{z} + g_2}. \] (7)

We then check the execution of the necessary condition for the singular control to be optimal (Kelley condition) from [14]. Therefore the optimal control \( u_*(t) \) may contain singular arcs. It follows from (4) that the following statement is valid.

**Theorem 6.** The optimal control \( u_*(t) \) is a piecewise constant function taking values \( \{u_1, \hat{u}, u_2\} \), if \( \hat{u} \in (u_1, u_2) \), and values \( \{u_1, u_2\} \) if \( \hat{u} \notin (u_1, u_2) \).

5. Hamiltonian system and its properties. In order to find the type of optimal control \( u_*(t) \) and to estimate the number of its switchings, we need to investigate the two point boundary value problem of the Maximum Principle (1),(3),(4) in its original form. Unfortunately, this is not analytically possible. Further we will demonstrate how knowledge of the specifics of the problem allows us to reduce this very complex problem (1),(3),(4) to an equivalent, but relatively easier problem.

First, for \( z \geq 0 \) we may introduce the monotonically increasing function
\[ w(z) = \frac{(\phi_1 + \phi_2 z^\gamma)z}{g_1 z + g_2}, \]
for which the following relationships are valid:
\[ w(0) = 0, \quad \lim_{z \to +\infty} w(z) = +\infty. \]

Let \( z_1, z_2 \) be such that \( 0 < z_1 < z_2 \) and \( w(z_1) = u_1, \ w(z_2) = u_2 \). It also follows from (7) that \( w(\hat{z}) = \hat{u} \) and the inequalities below that:
\[ \begin{cases} \hat{z} \leq z_1 < z_2, & \text{if } \hat{u} \leq u_1 < u_2, \\ z_1 < \hat{z} < z_2, & \text{if } u_1 < \hat{u} < u_2, \\ z_1 < z_2 \leq \hat{z}, & \text{if } u_1 < u_2 \leq \hat{u}. \end{cases} \] (8)
Taking into account the variables (5), inequalities (8), the representation of the switching function \( L(t) \) and properties of the function \( w(z) \), the two point boundary value problem of the Maximum Principle (1),(3),(4) can be rewritten as

\[
\begin{align*}
\dot{z}(t) &= - (\phi_1 + \phi_2 z^\gamma(t))z(t) + (g_1 z(t) + g_2)u(z(t), \eta(t)), \quad t \in [0, T], \\
\dot{\eta}(t) &= (\phi_1 + (1 - \gamma)\phi_2 z^\gamma(t))\eta(t) + g_2 u(z(t), \eta(t)) \left( \eta(t) - \frac{g_1}{g_2} \right) - \frac{\gamma \phi_2}{z^{1-\gamma(t)}}, \\
z(0) &= z_0, \quad \eta(T) = 0; \quad z_0 = \frac{x_0}{x_1^0},
\end{align*}
\]

where

\[
u(z, \eta) = \begin{cases} u_1, & \text{if } z \leq z_1 \\ w(z), & \text{if } z_1 < z < z_2 \\ u_2, & \text{if } z \geq z_2 \end{cases}, \quad \text{if } \eta < \frac{g_1}{g_2}.
\]

(10)

For the variable \( x_1^0(t) \), we have the Cauchy problem

\[
\begin{align*}
x_1^0(t) &= x_0^0, \\
x_1^0(0) &= x_1^0.
\end{align*}
\]

The system of equations (9) and relationship (10) form the Hamiltonian system. For its analysis we will use the method of phase diagrams described in [15].

Above the line \( \eta = \frac{g_1}{g_2} \) all trajectories of the system of equations (9) are trajectories corresponding to the control \( u(z, \eta) = u_1 \). Below the line \( \eta = \frac{g_1}{g_2} \) trajectories correspond to the control \( u(z, \eta) = u_2 \). The line \( \eta = \frac{g_1}{g_2} \) by itself is the so-called the "sewing line", but not a trajectory of the system (9). Moreover, trajectories coming close to the line \( \eta = \frac{g_1}{g_2} \) either go through it, changing the value of control in accordance with (10) from \( u_2 \) into \( u_1 \) or from \( u_1 \) into \( u_2 \), or stay tangent to this line.

Next, we will divide the first quadrant into several regions such that in some regions the right hand sides of the equations of the system (9) is positive and in others it is negative or zero.

Investigating further, let us introduce the following sets:

\[
G = \{(z, \eta)^\top \in \mathbb{R}^2 : z > 0, \eta \geq 0\}, \quad P_0^0 = \{(z, \eta)^\top \in G : z_1 \leq z \leq z_2, \eta = \frac{g_1}{g_2}\},
\]

\[
P_0^1 = \{(z, \eta)^\top \in G : z = z_1, \eta < \frac{g_1}{g_2}\}, \quad P_0^2 = \{(z, \eta)^\top \in G : z = z_2, \eta > \frac{g_1}{g_2}\}.
\]

The line \( \eta = \frac{g_1}{g_2} \) divides set \( G \) into two subsets. For the first subset, \( \eta < \frac{g_1}{g_2} \), the control \( u(z, \eta) \) takes the value \( u_1 \), and the first equation of system (9) can be written as

\[
\dot{z}(t) = -(g_1 z(t) + g_2)(w(z(t)) - w(z_1)).
\]

(11)

For the second subset \( \eta > \frac{g_1}{g_2} \), the control \( u(z, \eta) \) takes value \( u_2 \) and the first equation of system (9) can be written as

\[
\dot{z}(t) = -(g_1 z(t) + g_2)(w(z(t)) - w(z_2)).
\]

(12)
From equations (11), (12), and relationship (10) we conclude that the right hand side of the first equation of system (9) is zero in the region

\[ P_0 = P_0^1 \cup P_0^2 \cup P_0^3. \]

It is positive in the region

\[ P_+ = \left\{ (z, \eta)^T \in G : z < z_1, \eta \leq \frac{g_1}{g_2} \right\} \cup \left\{ (z, \eta)^T \in G : z < z_2, \eta > \frac{g_1}{g_2} \right\}, \]

and negative in the region

\[ P_- = \left\{ (z, \eta)^T \in G : z > z_1, \eta < \frac{g_1}{g_2} \right\} \cup \left\{ (z, \eta)^T \in G : z > z_2, \eta \geq \frac{g_1}{g_2} \right\}. \]

Now we will define similar regions for the second equation of system (9). Let us rewrite it for the piecewise constant control \( u(t) = u(z(t), \eta(t)) \) as

\[ \dot{\eta}(t) = g_2 u(t) \eta^2(t) + (\phi_1 + (1 - \gamma) \phi_2 z^\gamma(t) - g_1 u(t)) \eta(t) - \frac{\gamma \phi_2}{z^{1 - \gamma}(t)}. \] (13)

Denoting

\[ a_u(z) = \phi_1 + (1 - \gamma) \phi_2 z^\gamma - g_1 u, \quad b_u(z) = \frac{\gamma \phi_2 g_2 u}{z^{1 - \gamma}}, \quad d_u(z) = a_u^2(z) + 4 b_u(z), \]

we find that \( b_u(z) > 0 \) and \( d_u(z) > 0 \) for all \( z > 0 \) and \( u \in \{ u_1, u_2 \} \). Then the following functions are defined:

\[ h_u^0(z) = \frac{-a_u(z) - \sqrt{d_u(z)}}{2 g_2 u}, \quad h_u(z) = \frac{-a_u(z) + \sqrt{d_u(z)}}{2 g_2 u}, \]

and the right hand side of (13) can be factored as

\[ \dot{\eta}(t) = g_2 u(t) (\eta(t) - h_u^0(z(t))) \cdot (\eta(t) - h_u(z(t))). \] (14)

Notice that in the region \( G \), the first factor of the right hand side is always positive.

Further, for all \( z > 0 \) and \( u \in \{ u_1, u_2 \} \) the function \( h_u(z) \) is monotonically decreasing and the following relationships are valid:

\[ \lim_{z \rightarrow +0} h_u(z) = +\infty, \quad \lim_{z \rightarrow +\infty} h_u(z) = 0, \quad h_u(z) = \frac{g_1}{g_2} = h_{u_1}(z). \]

The later relationships allow us to define the continuous monotonically decreasing function in the region \( G \),

\[ h(z) = \begin{cases} h_{u_1}(z), & \text{if } z < \hat{z}, \\ \frac{g_1}{g_2}, & \text{if } z = \hat{z}, \\ h_{u_2}(z), & \text{if } z > \hat{z}, \end{cases} \]

for which the following relationships hold:

\[ \lim_{z \rightarrow +0} h(z) = +\infty, \quad \lim_{z \rightarrow +\infty} h(z) = 0. \]

Moreover, from (14) we conclude that the right hand side of the second equation of system (9) becomes zero in the region

\[ Q_0 = \{(z, \eta)^T \in G : \eta = h(z)\}. \]

It is positive in the region

\[ Q_+ = \{(z, \eta)^T \in G : \eta > h(z)\}, \]
and negative in the region
\[ Q_− = \{(z, η)^T ∈ G : η < h(z)\} . \]

We thus obtain the following description of the vector field of the Hamiltonian system (9), (10).

**Theorem 7.** The vector field of the Hamiltonian system (9), (10) in the region \( G \) is positive in both coordinates \( z \) and \( η \) in the region \( R_{++} = P_+ \cap Q_+ \), negative in both coordinates \( z \) and \( η \) in the region \( R_{--} = P_- \cap Q_- \), is positive in \( z \) and negative in \( η \) in the region \( R_{+-} = P_+ \cap Q_- \), is negative in \( z \) and positive in \( η \) in the region \( R_{-+} = P_- \cap Q_+ \), becomes zero in coordinate \( z \) in the region \( P_0 \), and becomes zero in coordinate \( η \) in the region \( Q_0 \).

Obviously, the following relationship is valid:
\[ G = R_{++} \cup R_{+-} \cup R_{-+} \cup R_{--} \cup P_0 \cup Q_0 . \]

In addition, sets \( P_0^1 \) and \( P_0^2 \) form trajectories of the system of equations (9). Now let us study the rest point \((z_*, η_*)^T\) of the Hamiltonian system (9), (10). It is the common point of sets \( P_0 \) and \( Q_0 \).

**Theorem 8.** For the Hamiltonian system (9), (10) the following statements are valid:

1. If \( u < u_1 < u_2 \), then the system of equations (9) has a unique rest point \((z_*, η_*)^T\) under the control \( u(z, η) = u_1 \) for which \( z_* = z_1, η_* = h(z_1) \), and \( η_* < \frac{u_1}{g_z} \).
2. If \( u_1 < u < u_2 \), then the system of equations (9) has a unique rest point \((z_*, η_*)^T\) under the control \( u(z, η) = u_2 \) such that \( z_* = z_2, η_* = h(z_2) \), and \( η_* > \frac{u_2}{g_z} \).
3. If \( u_1 < \hat{u} < u_2 \), then the Hamiltonian system (9), (10) has no rest point.
4. If \( \hat{u} = u_1 < u_2 \), then there is a unique point \((z_*, η_*)^T\) for which \( z_* = z_1 = \hat{z} \) and \( η_* = h(\hat{z}) = \frac{u_1}{g_z} \). It lies on the sewing line \( η = \frac{u_1}{g_z} \) and is the rest point of the system of equations (9) only under control \( u(z, η) = u_1 \). Trajectories of the system of equations (9) under control \( u(z, η) = u_2 \) reach \((z_*, η_*)^T\) only at a finite moment \( t \).
5. If \( u_1 < u_2 = \hat{u} \), then there is unique point \((z_*, η_*)^T\), such that \( z_* = z_2 = \hat{z} \) and \( η_* = h(\hat{z}) = \frac{u_2}{g_z} \). It lies on the sewing line \( η = \frac{u_2}{g_z} \) and is the rest point of the system of equations (9) only under control \( u(z, η) = u_2 \). Trajectories of the system of equations (9) under control \( u(z, η) = u_1 \) reach \((z_*, η_*)^T\) only at finite moment \( t \).

Theorem 7 and Theorem 8 completely describe the qualitative behavior of solutions of the Hamiltonian system (9), (10) in region \( G \).

The most interesting case is where \( u_1 < \hat{u} < u_2 \) because the optimal control \( u_*(t) \) of the problem (1), (2) may contain singular arcs. This particular case will be considered below. For all other cases only the final results will be stated. The vector field of the Hamiltonian system (9), (10) in this case is shown in Figure 1. In what follows, let \( \overline{E} \) denote the closure of set \( E \) in \( \mathbb{R}^2 \).

**Theorem 9.** Let \((z(t), η(t))^T\) be the solution of the Hamiltonian system (9), (10) in region \( G \) which is nonextendable to the right, \( Δ \) is the interval on which it is defined, and \( t_* ∈ Δ \). Then the following statements hold true:

1. If \((z(t_*), η(t_*))^T \in R_{++}, \) then the interval \( Δ \) is unbounded, \((z(t), η(t))^T \in R_{++} \) for all \( t \in Δ \cap (t_*, +∞) \) and the relationships
   \[ \lim_{t \to +∞} z(t) = z_2, \lim_{t \to +∞} η(t) = +∞, \] (15)
are valid.
2. If \((z(t_*), \eta(t_*))^\top \in R_-,\) then the interval \(\Delta\) is bounded, \((z(t), \eta(t))^\top \in R_-\) for all \(t \in \Delta \cap (t_*, +\infty)\) and \(\eta(\theta) = 0\), where \(\theta = \sup_{t \in \Delta \cap \{t\}}\).

3. If \((z(t_*), \eta(t_*))^\top \in R_+,\) then one of the following cases occurs :
   a) the interval \(\Delta\) is unbounded, \((z(t), \eta(t))^\top \in R_+\) for all \(t \in \Delta \cap (t_*, +\infty)\) for some \(t^*\) in \(\Delta \cap (t_*, +\infty)\) and relationships (15) are valid.
   b) the interval \(\Delta\) is bounded, \((z(t), \eta(t))^\top \in R_-\) for all \(t \in \Delta \cap (t_*, +\infty)\) for some \(t^*\) in \(\Delta \cap (t_*, +\infty)\) and \(\eta(\theta) = 0\), where \(\theta = \sup_{t \in \Delta \cap \{t\}}\).
   c) the interval \(\Delta\) is bounded, \((z(t), \eta(t))^\top \in R_+\) for all \(t \in \Delta \cap (t_*, +\infty)\) and \(\eta(\theta) = 0\), where \(\theta = \sup_{t \in \Delta \cap \{t\}}\).
   d) the interval \(\Delta\) is bounded, \((z(t), \eta(t))^\top \in R_-\) for all \(t \in \Delta \cap (t_*, +\infty)\) and \(z(\theta) = \hat{\theta}, \eta(\theta) = \frac{\hat{\theta}}{92},\) where \(\theta = \sup_{t \in \Delta \cap \{t\}}\).

4. If \((z(t_*), \eta(t_*))^\top \in R_+,\) then one of the following cases occurs :
   a) the interval \(\Delta\) is unbounded, \((z(t), \eta(t))^\top \in R_+\) for all \(t \in \Delta \cap (t_*, +\infty)\) and relationships (15) are valid.
   b) the interval \(\Delta\) is bounded, \((z(t), \eta(t))^\top \in R_-\) for all \(t \in \Delta \cap (t_*, +\infty)\) for some \(t^*\) in \(\Delta \cap (t_*, +\infty)\) and \(\eta(\theta) = 0\), where \(\theta = \sup_{t \in \Delta \cap \{t\}}\).
   c) the interval \(\Delta\) is bounded, \((z(t), \eta(t))^\top \in R_+\) for all \(t \in \Delta \cap (t_*, +\infty)\) and relationships (15) are valid.
   d) the interval \(\Delta\) is bounded, \((z(t), \eta(t))^\top \in R_-\) for all \(t \in \Delta \cap (t_*, +\infty)\) and \(z(\theta) = \hat{\theta}, \eta(\theta) = \frac{\hat{\theta}}{92},\) where \(\theta = \sup_{t \in \Delta \cap \{t\}}\).

5. If \((z(t_*), \eta(t_*))^\top \in Q_0,\) then one of the following cases occurs :
   a) if \(z(t_*) < \hat{\theta},\) then the interval \(\Delta\) is bounded, \((z(t), \eta(t))^\top \in R_-\) for all \(t \in \Delta \cap (t_*, +\infty)\) and \(\eta(\theta) = 0\), where \(\theta = \sup_{t \in \Delta \cap \{t\}}\).
   c) if \(z(t_*) = \hat{\theta},\) then the interval \(\Delta\) can be either bounded, \((z(t), \eta(t))^\top \in R_-\) for all \(t \in \Delta \cap (t_*, +\infty)\) and \(\eta(\theta) = 0\), where \(\theta = \sup_{t \in \Delta \cap \{t\}}\) or unbounded, \((z(t), \eta(t))^\top \in R_+\) for all \(t \in \Delta \cap (t_*, +\infty)\) and relationships (15) are valid.

6. If \((z(t_*), \eta(t_*))^\top \in P_0,\) then one of the following cases occurs :
   a) if \((z(t_*), \eta(t_*))^\top \in P_0^1,\) then the interval \(\Delta\) is bounded, \((z(t), \eta(t))^\top \in P_0^1\) for all \(t \in \Delta \cap (t_*, +\infty)\) and \(\eta(\theta) = 0\), where \(\theta = \sup_{t \in \Delta \cap \{t\}}\).
   b) if \((z(t_*), \eta(t_*))^\top \in P_0^2,\) then the interval \(\Delta\) is unbounded, \((z(t), \eta(t))^\top \in P_0^2\) for all \(t \in \Delta \cap (t_*, +\infty)\) and relationships (15) are valid.

Theorem 10. For the solution \((z(t), \eta(t))^\top\) of the two point boundary value problem of the Maximum Principle (9),(10) the optimal control \(u_*(t) = u(z(t), \eta(t))\) defined by relationship (10) is either a constant function

\[ u_*(t) = u_1, \text{ for all } t \in [0, T], \]
or a piecewise constant function with one switching
\[
u_*(t) = \begin{cases} 
u_1, & \text{if } t \in [0, \tau], \\ \nu_2, & \text{if } t \in (\tau, T], \end{cases}
\]
where \( \tau \in (0, T) \) is the moment of switching.

or piecewise constant function with two switchings and containing singular arcs of one of the types:
\[
u_*(t) = \begin{cases} \nu_1, & \text{if } t \in [0, \tau_1], \\ \hat{u}, & \text{if } t \in (\tau_1, \tau_2], \\ \nu_1, & \text{if } t \in (\tau_2, T], \end{cases}
\]
\[
u_*(t) = \begin{cases} \nu_2, & \text{if } t \in [0, \tau_1], \\ \hat{u}, & \text{if } t \in (\tau_1, \tau_2], \\ \nu_1, & \text{if } t \in (\tau_2, T], \end{cases}
\]
where \( \tau_1, \tau_2 \in (0, T) \) are the moments of switching.

This case is shown by Figure 1. If a trajectory \((z(t), \eta(t))\) of the system (9) comes to the point of intersection of the line \(h(z)\) (appears like a hyperbola on the graph) and horizontal line \(\eta = \frac{2z}{g_2}\), then the optimal control \(u_*(t) = u(z(t), \eta(t))\) during time interval \([0, \tau_1]\) changes from \(u_1\) or \(u_2\) to the singular control \(\hat{u}\). The system (9) will stay at this location from time \(\tau_1\) to time \(\tau_2\). Then the system (9) will leave this point and start moving towards line \(\eta = 0\). Finally, the control \(u_*(t)\) will take value \(u_1\) and stay at this value on the interval \((\tau_2, T]\).

For all other cases we have the following results. If \(\hat{u} \leq u_1 < u_2\) or \(u_1 < u_2 = \hat{u}\), the optimal control \(u_*(t)\) of the problem (1),(2) can be either a constant function of type (16) or a piecewise constant function with one switching of type (17).

If \(u_1 < u_2 < \hat{u}\), then the optimal control \(u_*(t)\) of the problem (1),(2) can be either a constant function of type (16), a piecewise constant function with one switching of type (17), or a piecewise constant function with two switchings of type
\[
u_*(t) = \begin{cases} \nu_1, & \text{if } t \in [0, \tau_1], \\ \nu_2, & \text{if } t \in (\tau_1, \tau_2], \\ \nu_1, & \text{if } t \in (\tau_2, T], \end{cases}
\]
where \(\tau_1, \tau_2 \in (0, T)\) are the moments of switching.

6. Computer modeling. System (1) cannot be integrated analytically under piecewise constant controls. However, for any given parameters of the model, the optimal trajectory can be found numerically using methods of finite dimensional optimization ([17]). The moments of switching \(\tau_1, \tau_2 \in (0, T)\) of the optimal control \(u_*(t)\) are parameters of optimization. First, for the parameters \(\phi_1, \phi_2, \gamma, g_1, g_2, T, x_1^0, x_2^0\) of the system (1) we estimate the value of \(\hat{z}\) and \(\hat{u}\) from Section 4. Based on relationships between \(u_1, u_2, \hat{u}\) and \(\hat{u}\) we determine the possible types of the optimal controls (16),(17),(18),(19) or (20), and then system (1) is integrated numerically with the selected type of optimal control \(u_*(t)\). Methods of finite dimensional optimization are used to obtain the maximum value of the corresponding functional (2).

We consider the situation when \(\phi_1 = 0.14, \phi_2 = 0.1, \gamma = 0.5, g_1 = 0.67, g_2 = 1.0, T = 5\) years, \(x_1^0 = 427.22, x_2^0 = 157.1, u_1 = 0.01, u_2 = 1.0\), that model the Sony Corporation in FY 1980. Variables \(x_1\) and \(x_2\) are measured in billion of Yen. The
maximum value of the functional $J(u)$ is $J_* = 702.7$ and comes from control of type (18) with moments of switching $\tau_1 = 2.5$ and $\tau_2 = 3$ (see Figure 2).

Therefore, in order to maximize production, the company had to invest in technology at the minimum rate $u_1$ during 2.5 years, then switch to a singular regime at the intensity $\dot{u} = 0.073$, and stay there for 0.5 year, then again continue at the rate $u_1$. The optimal trajectory is shown on Figure 3. Since the value of the total production, $x_2(5) = 702.7$, is in agreement with the actual production for the Sony Corporation over a five year period (1980-1985) ([18]), it is suggested that system (1) may be a valid economic growth model and that functional (2) may be useful for the measurement of the effectiveness of economic growth.
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